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Numerical solutions to Helmholtz equation of anisotropic functionally graded materials

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Abstract. In this paper, interior 2D-BVPs for anisotropic FGMs governed by the Helmholtz equation with Dirichlet and Neumann boundary conditions are considered. The governing equation involves diffusivity and wave number coefficients which are spatially varying. The anisotropy of the material is presented in the diffusivity coefficient. And the inhomogeneity is described by both diffusivity and wave number. Three types of the gradation function considered are quadratic, exponential and trigonometric functions. A technique of transforming the variable coefficient governing equation to a constant coefficient equation is utilized for deriving a boundary integral equation. And a standard BEM is constructed from the boundary integral equation to find numerical solutions. Some particular examples of BVPs are solved to illustrate the application of the BEM. The results show the accuracy of the BEM solutions, especially for large wave numbers. They also show coherence between the flow vectors and scattering solutions, and the effect of the anisotropy and inhomogeneity of the material on the BEM solutions.

1. Introduction

A functionally graded material/medium (FGM) is commonly defined as an inhomogeneous material having a specific property such as thermal conductivity, hardness, toughness, ductility, corrosion resistance, etc. that changes spatially in a continuous fashion. Nowadays FGM has become an important topic and numerous studies on FGM for a variety of applications including wave propagation have been reported (see for example [1], [2]).

The boundary element method (BEM) has been successfully used for solving many types of problems of either homogeneous or inhomogeneous, and either isotropic or anisotropic materials. Paper [3] reported a work on unsteady transport problems in homogeneous media. For FGMs there are two main techniques usually used. The first one uses a technique of deriving a relevant Green function or fundamental solution to the FGM problem. Cheng [4] had applied this technique. The second technique is by transforming the variable coefficient governing equation to a constant coefficient equation. Some progress on using the second technique has been made. For anisotropic FGM, papers [5, 6, 7, 8, 9, 10] reported works on a different class of equations for problems of anisotropic FGMs.

The Helmholtz equation is usually used to mathematically model physical processes such as wave propagation, electromagnetic radiation and seismology. The Helmholtz equation demonstrates the quality of linear wave problem to depict scattering phenomenon, which



has great importance in physics and engineering ([11]). A number of studies dealing with the Helmholtz equation have been reported, however most of the studies deal with isotropic homogeneous materials. In [11] the Helmholtz with large wave number (wave numbers ≤ 10 taken in the examples) was considered for isotropic homogeneous materials. The weak Galerkin mixed finite element method (FEM) used was found to be robust even when the wave number is large. The paper [12] utilized a Galerkin BEM to find numerical solution to exterior problems of isotropic homogeneous media governed by 2-D Helmholtz equation with arbitrary wave number (wave numbers ≤ 5.1 taken in the examples) and found that the scheme is practical and effective. Paper [13] also deals with Helmholtz equation for isotropic homogeneous media in connection with finding analytical integration of the weakly singular integrals in a boundary element analysis. Moreover, in [14] Helmholtz problems of isotropic homogeneous media were solved using the BEM of direct radial basis function interpolation. In addition, papers [15, 16, 17, 18] also deal with Helmholtz equation for isotropic homogeneous media. Furthermore, in [19] interior and exterior problems of Helmholtz equation were solved using a spectral FEM for isotropic inhomogeneous materials, in [20] the Helmholtz equation was solved using a finite difference method (FDM) for anisotropic homogeneous media, and in [21] the Helmholtz equation was solved using BEM for anisotropic homogeneous media.

This paper discusses derivation of a boundary integral equation for numerically solving 2D interior boundary value problems governed by the Helmholtz type equation for anisotropic FGMs of the form

$$\frac{\partial}{\partial x_i} \left[\lambda_{ij}(x_1, x_2) \frac{\partial \phi(x_1, x_2)}{\partial x_j} \right] + \beta^2(x_1, x_2) \phi(x_1, x_2) = 0 \quad (1)$$

where the coefficients λ_{ij} and β^2 depend on x_1 and x_2 and the repeated summation convention (summing from 1 to 2) is employed. As for the case $\beta^2 = 0$ a study has been done in [5], this paper will be restricted only for the case $\beta^2 > 0$.

The technique of transforming (1) to a constant coefficient equation will be used for obtaining a boundary integral equation for the solution of (1). It is necessary to place some constraint on the class of coefficients λ_{ij} and β for which the solution obtained is valid. Throughout the paper, the analysis used is purely mathematical; to develop a BEM for obtaining the numerical solution of BVPs of FGMs governed by equation (1) is the main purpose. Additional aims are to study the feasibility of BEM in solving such kind of problems especially for large wave numbers, and also to see the impact of anisotropy and inhomogeneity of materials on the scattering solutions.

2. The boundary value problem (BVP)

The BVP is restricted to interior two-dimensional (2D) problem with boundary conditions of type Dirichlet or Neumann. Referred to a Cartesian frame Ox_1x_2 a solution to (1) is sought which is valid in a region Ω in R^2 with boundary $\partial\Omega$ consisting of a number of piecewise continuous curves. On $\partial\Omega$ either $\phi(\mathbf{x})$ or $P(\mathbf{x})$ is specified, where

$$P(\mathbf{x}) = \lambda_{ij} (\partial\phi/\partial x_j) n_i \quad (2)$$

$\mathbf{x} = (x_1, x_2)$ and $\mathbf{n} = (n_1, n_2)$ is the normal vector pointing out on the boundary $\partial\Omega$. For equation (1) to be an elliptic partial differential equation throughout Ω , the matrix of coefficients $[\lambda_{ij}]$ is required to be a symmetric positive definite matrix. The coefficients λ_{ij} and β are also required to be twice differentiable functions.

A boundary integral equation will be sought, from which numerical values of the dependent variables ϕ and its derivatives may be obtained for all points in Ω . The analysis is in general applicable for anisotropic media but it is not excepted to isotropic materials. The analysis also applies for the case of isotropic media that is when $\lambda_{11} = \lambda_{22}$ and $\lambda_{12} = 0$.

3. The boundary integral equation

The boundary integral equation is derived by transforming the variable coefficient equation (1) to a constant coefficient equation. We restrict the coefficients λ_{ij} and β to be of the form

$$\lambda_{ij}(\mathbf{x}) = \bar{\lambda}_{ij}g(\mathbf{x}) \quad (3)$$

$$\beta^2(\mathbf{x}) = \bar{\beta}^2g(\mathbf{x}) \quad (4)$$

where $g(\mathbf{x})$ is a differentiable function and $\bar{\lambda}_{ij}$ and $\bar{\beta}^2$ are constant. Substitution of (3) and (4) into (1) gives

$$\bar{\lambda}_{ij} \frac{\partial}{\partial x_i} \left(g \frac{\partial \phi}{\partial x_j} \right) + \bar{\beta}^2 g \phi = 0 \quad (5)$$

Assume

$$\phi(\mathbf{x}) = g^{-1/2}(\mathbf{x}) \psi(\mathbf{x}) \quad (6)$$

therefore equation (5) can be written as

$$\bar{\lambda}_{ij} \frac{\partial}{\partial x_i} \left[g \frac{\partial (g^{-1/2} \psi)}{\partial x_j} \right] + \bar{\beta}^2 g^{1/2} \psi = 0$$

which can be further written as

$$\bar{\lambda}_{ij} \left[\left(\frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j} \right) \psi + g^{1/2} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right] + \bar{\beta}^2 g^{1/2} \psi = 0 \quad (7)$$

Use of the identity

$$\frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} = -\frac{1}{4} g^{-3/2} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} + \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial x_i \partial x_j}$$

allows equation (7) to be written in the form

$$g^{1/2} \bar{\lambda}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} - \psi \bar{\lambda}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + \bar{\beta}^2 g^{1/2} \psi = 0 \quad (8)$$

So that if g satisfies

$$\bar{\lambda}_{ij} \frac{\partial^2 g^{1/2}}{\partial x_i \partial x_j} + k g^{1/2} = 0 \quad (9)$$

where k is a constant, then the transformation (6) brings the variable coefficients equation (5) into a constant coefficients equation

$$\bar{\lambda}_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} + (k + \bar{\beta}^2) \psi = 0 \quad (10)$$

Moreover, substitution of (3) and (6) into (2) gives

$$P = -P_g \psi + P_\psi g^{1/2} \quad (11)$$

where $P_g(\mathbf{x}) = \bar{\lambda}_{ij} (\partial g^{1/2} / \partial x_j) n_i$ and $P_\psi(\mathbf{x}) = \bar{\lambda}_{ij} (\partial \psi / \partial x_j) n_i$

Three possible multi parameter function $g(\mathbf{x})$ satisfying (9) are $g(\mathbf{x}) = [A(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2)]^2$ for which $k = 0$, $g(\mathbf{x}) = [A \exp(\alpha_m x_m)]^2$ for which $k < 0$ and $\bar{\lambda}_{ij} \alpha_i \alpha_j = -k$, and $g(\mathbf{x}) = \{A [\cos(\alpha_m x_m) + \sin(\alpha_m x_m)]\}^2$ for which $k > 0$ and $\bar{\lambda}_{ij} \alpha_i \alpha_j = k$. When the material under consideration is a layered material consisting of several layers where

each layer is a specific type of material of specific constant coefficients $\bar{\lambda}_{ij}$ and $\bar{\beta}^2$ then the discrete variation of the constant coefficients from layer to layer may certainly accommodate the determination of a continuous variation of the variable coefficients $\lambda_{ij}(\mathbf{x})$ and $\beta^2(\mathbf{x})$ by interpolation, that is to determine the parameters α_m of function $g(\mathbf{x})$.

An integral equation for (10) is

$$\eta(\mathbf{x}_0) \psi(\mathbf{x}_0) = \int_{\partial\Omega} [\Gamma(\mathbf{x}, \mathbf{x}_0) \psi(\mathbf{x}) - \Phi(\mathbf{x}, \mathbf{x}_0) P_\psi(\mathbf{x})] ds(\mathbf{x}) \tag{12}$$

where $\mathbf{x}_0 = (a, b)$, $\eta = 0$ if $(a, b) \notin \Omega \cup \partial\Omega$, $\eta = 1$ if (a, b) lies inside the domain Ω , $\eta = \frac{1}{2}$ if (a, b) is on the boundary $\partial\Omega$ given that $\partial\Omega$ has a continuously turning tangent at (a, b) . The function Φ in (12) is called the fundamental solution, which is any solution of the equation $\bar{\lambda}_{ij}(\partial^2\Phi/\partial x_i\partial x_j) + (k + \bar{\beta}^2)\Phi = \delta(\mathbf{x} - \mathbf{x}_0)$ and the Γ is defined as $\Gamma(\mathbf{x}, \mathbf{x}_0) = \bar{\lambda}_{ij}[\partial\Phi(\mathbf{x}, \mathbf{x}_0)/\partial x_j]n_i$ where δ denotes the Dirac delta function. Following Azis [22], for 2-D problems Φ and Γ are given by

$$\Phi(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \ln R & \text{if } k + \bar{\beta}^2 = 0 \\ \frac{iK}{4} H_0^{(2)}(\omega R) & \text{if } k + \bar{\beta}^2 > 0 \\ -\frac{K}{2\pi} K_0(\omega R) & \text{if } k + \bar{\beta}^2 < 0 \end{cases} \tag{13}$$

$$\Gamma(\mathbf{x}, \mathbf{x}_0) = \begin{cases} \frac{K}{2\pi} \frac{1}{R} \bar{\lambda}_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } k + \bar{\beta}^2 = 0 \\ -\frac{iK\omega}{4} H_1^{(2)}(\omega R) \bar{\lambda}_{ij} \frac{\partial R}{\partial x_j} n_i & \text{if } k + \bar{\beta}^2 > 0 \\ \frac{K\omega}{2\pi} K_1(\omega R) \lambda_{ij}^{(0)} \frac{\partial R}{\partial x_j} n_i & \text{if } k + \bar{\beta}^2 < 0 \end{cases} \tag{14}$$

where $K = \bar{\tau}/\zeta$, $\omega = \sqrt{|\bar{\beta}^2/\zeta|}$, $\zeta = [\bar{\lambda}_{11} + 2\bar{\lambda}_{12}\bar{\tau} + \bar{\lambda}_{22}(\bar{\tau}^2 + \bar{\tau}^2)]/2$, $R = \sqrt{(\hat{x}_1 - \hat{a})^2 + (\hat{x}_2 - \hat{b})^2}$, $\hat{x}_1 = x_1 + \bar{\tau}x_2$, $\hat{a} = a + \bar{\tau}b$, $\hat{x}_2 = \bar{\tau}x_2$ and $\hat{b} = \bar{\tau}b$ where $\bar{\tau}$ and $\bar{\tau}$ are respectively the real and the positive imaginary parts of the complex root τ of the quadratic $\bar{\lambda}_{11} + 2\bar{\lambda}_{12}\bar{\tau} + \bar{\lambda}_{22}\bar{\tau}^2 = 0$ and $H_0^{(2)}$, $H_1^{(2)}$ denote the Hankel function of second kind and order zero and order one respectively, K_0 , K_1 denote the modified Bessel function of order zero and order one respectively and i represents the square root of minus one. The derivatives $\partial R/\partial x_j$ necessary for the calculation of the Γ in (14) are given by $\partial R/\partial x_1 = (\hat{x}_1 - \hat{a})/R$ and $\partial R/\partial x_2 = [\bar{\tau}(\hat{x}_1 - \hat{a}) + \bar{\tau}(\hat{x}_2 - \hat{b})]/R$. Use of (6) and (11) in (12) yields

$$\eta(\mathbf{x}_0) g^{1/2}(\mathbf{x}_0) \phi(\mathbf{x}_0) = \int_{\partial\Omega} \left\{ \left[g^{1/2}(\mathbf{x}) \Gamma(\mathbf{x}, \mathbf{x}_0) - P_g(\mathbf{x}) \Phi(\mathbf{x}, \mathbf{x}_0) \right] \phi(\mathbf{x}) - \left[g^{-1/2}(\mathbf{x}) \Phi(\mathbf{x}, \mathbf{x}_0) \right] P(\mathbf{x}) \right\} ds(\mathbf{x}) \tag{15}$$

Equation (15) provides a boundary integral equation which is the starting point of BEM construction for determining the numerical solutions of ϕ and its derivatives at all points of Ω .

4. Discretisation

Let the boundary $\partial\Omega$ be discretised into J elements $\partial\Omega_j \equiv [\mathbf{q}_{j-1}, \mathbf{q}_j]$ for $j = 1, 2, 3, \dots, J$ where \mathbf{q}_{j-1} and \mathbf{q}_j are the endpoints of the element $\partial\Omega_j$. It is assumed that ϕ and P are constant along each boundary element $\partial\Omega_j$ taking on their values at the mid point $\hat{\mathbf{q}}_j = (\mathbf{q}_{j-1} + \mathbf{q}_j)/2$.

then the discretised form of (15) may be written as

$$\eta(\mathbf{x}_0) g^{1/2}(\mathbf{x}_0) \phi(\mathbf{x}_0) = \sum_{j=1}^J \left\{ -P(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} [g^{-1/2}(\mathbf{x}) \Phi(\mathbf{x}, \mathbf{x}_0)] ds(\mathbf{x}) + \phi(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} [g^{1/2}(\mathbf{x}) \Gamma(\mathbf{x}, \mathbf{x}_0) - P_g(\mathbf{x}) \Phi(\mathbf{x}, \mathbf{x}_0)] ds(\mathbf{x}) \right\} \quad (16)$$

The integral equation (16) is used to find the boundary unknowns $\phi(\mathbf{x})$ on $\partial\Omega_2$ and $P(\mathbf{x})$ on $\partial\Omega_1$ using the given boundary data $\phi(\mathbf{x})$ on $\partial\Omega_1$ and $P(\mathbf{x})$ on $\partial\Omega_2$. Then the solutions $\phi(\mathbf{x})$ and its derivatives in the domain Ω are evaluated using the complete boundary data $\phi(\mathbf{x})$ and $P(\mathbf{x})$ on $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$.

If the source point \mathbf{x}_0 lies on the boundary $\partial\Omega$ (thus $\eta(\mathbf{x}_0) = \frac{1}{2}$), say \mathbf{x}_0 lies on the boundary element $\partial\Omega_i$ ($i = 1, 2, \dots, J$) so that \mathbf{x}_0 is the mid-point $\hat{\mathbf{q}}_i$ of $\partial\Omega_i$, then equation (16) can be written as

$$\frac{1}{2} g^{1/2}(\hat{\mathbf{q}}_i) \phi(\hat{\mathbf{q}}_i) = \sum_{j=1}^J \left\{ -P(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} [g^{-1/2}(\mathbf{x}) \Phi(\mathbf{x}, \hat{\mathbf{q}}_i)] ds(\mathbf{x}) + \phi(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} [g^{1/2}(\mathbf{x}) \Gamma(\mathbf{x}, \hat{\mathbf{q}}_i) - P_g(\mathbf{x}) \Phi(\mathbf{x}, \hat{\mathbf{q}}_i)] ds(\mathbf{x}) \right\}$$

for $i = 1, 2, \dots, J$. This equation may be written in matrix form

$$\frac{1}{2} g_i^{1/2} \phi_i - \sum_{j=1}^J \hat{H}_{ij} \phi_j = - \sum_{j=1}^J G_{ij} P_j \quad (17)$$

where $g_i^{1/2} = g^{1/2}(\hat{\mathbf{q}}_i)$, $\phi_i = \phi(\hat{\mathbf{q}}_i)$, $P_j = P(\hat{\mathbf{q}}_j)$, and

$$\hat{H}_{ij} = \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} [g^{1/2}(\mathbf{x}) \Gamma(\mathbf{x}, \hat{\mathbf{q}}_i) - P_g(\mathbf{x}) \Phi(\mathbf{x}, \hat{\mathbf{q}}_i)] ds(\mathbf{x}) \quad (18)$$

$$G_{ij} = \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} [g^{-1/2}(\mathbf{x}) \Phi(\mathbf{x}, \hat{\mathbf{q}}_i)] ds(\mathbf{x}) \quad (19)$$

Evaluation of the integrals in equations (18) and (19) can be done numerically. And also the values of the modified Bessel function K_0 which is involved in the fundamental solutions Φ can be approached by its polynomial approximations (see Abramowitz and Stegun [23]).

In a simpler manner, equation (17) may be written as

$$\sum_{j=1}^J H_{ij} \phi_j = - \sum_{j=1}^J G_{ij} P_j \quad (20)$$

where

$$H_{ij} = \begin{cases} -\hat{H}_{ij} & \text{when } i \neq j \\ \frac{1}{2} g_i^{1/2} - \hat{H}_{ij} & \text{when } i = j \end{cases}$$

Equation (20) can be rearranged by putting the unknowns on the left hand side and all the knowns on the right hand side to obtain a $J \times J$ linear system of algebraic equations in the form

$$\mathbf{AX} = \mathbf{B} \quad (21)$$

where \mathbf{X} is the unknown matrix.

Once the unknowns ϕ and P on the boundary Γ are obtained from the equation (21), we can calculate the value of ϕ at any point \mathbf{x}_0 inside the domain Ω by using the equation (16). That is

$$\begin{aligned} \phi(\mathbf{x}_0) = & g^{-1/2}(\mathbf{x}_0) \sum_{j=1}^J \left\{ -P(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} \left[g^{-1/2}(\mathbf{x}) \Phi(\mathbf{x}, \mathbf{x}_0) \right] ds(\mathbf{x}) \right. \\ & \left. + \phi(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} \left[g^{1/2}(\mathbf{x}) \Gamma(\mathbf{x}, \mathbf{x}_0) - P_g(\mathbf{x}) \Phi(\mathbf{x}, \mathbf{x}_0) \right] ds(\mathbf{x}) \right\} \end{aligned}$$

We can also calculate the values of the derivatives $\partial\phi/\partial a$ and $\partial\phi/\partial b$ using the following equations

$$\begin{aligned} \frac{\partial\phi}{\partial a}(\mathbf{x}_0) = & g^{-1/2}(\mathbf{x}_0) \left\{ \sum_{j=1}^J \left\{ -P(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} \left[g^{-1/2}(\mathbf{x}) \frac{\partial\Phi(\mathbf{x}, \mathbf{x}_0)}{\partial a} \right] ds(\mathbf{x}) \right. \right. \\ & \left. \left. + \phi(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} \left[g^{1/2}(\mathbf{x}) \frac{\partial\Gamma(\mathbf{x}, \mathbf{x}_0)}{\partial a} - P_g(\mathbf{x}) \frac{\partial\Phi(\mathbf{x}, \mathbf{x}_0)}{\partial a} \right] ds(\mathbf{x}) \right\} \right. \\ & \left. - \phi(\mathbf{x}_0) \frac{\partial g^{1/2}(\mathbf{x}_0)}{\partial a} \right\} \end{aligned}$$

$$\begin{aligned} \frac{\partial\phi}{\partial b}(\mathbf{x}_0) = & g^{-1/2}(\mathbf{x}_0) \left\{ \sum_{j=1}^J \left\{ -P(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} \left[g^{-1/2}(\mathbf{x}) \frac{\partial\Phi(\mathbf{x}, \mathbf{x}_0)}{\partial b} \right] ds(\mathbf{x}) \right. \right. \\ & \left. \left. + \phi(\hat{\mathbf{q}}_j) \int_{\mathbf{q}_{j-1}}^{\mathbf{q}_j} \left[g^{1/2}(\mathbf{x}) \frac{\partial\Gamma(\mathbf{x}, \mathbf{x}_0)}{\partial b} - P_g(\mathbf{x}) \frac{\partial\Phi(\mathbf{x}, \mathbf{x}_0)}{\partial b} \right] ds(\mathbf{x}) \right\} \right. \\ & \left. - \phi(\mathbf{x}_0) \frac{\partial g^{1/2}(\mathbf{x}_0)}{\partial b} \right\} \end{aligned}$$

5. Numerical examples

To illustrate the use of BEM some examples of BVPs governed by (1) for FGMs are considered. For a simplicity, the domain Ω is taken to be a unit square for all BVPs (see Figure 1). Hankel and the modified Bessel functions in (13) and (14) are approximated by their ascending series, and the integral in (15) is evaluated using Bode's quadrature with 10 nodal points and error order $O(h^{11})$ (see Abramowitz and Stegun [23]). A number of elements of equal length on each side of the unit square are used. A FORTRAN script is developed to compute the solutions and a specific FORTRAN command is imposed to calculate the elapsed CPU computation time for obtaining the results.

5.1. Test problems

Some problems with analytical solutions will be considered. The aim is to evaluate the accuracy and efficiency of the BEM solutions. In addition to this, the impact of the wave number $\beta(\mathbf{x})$ on the accuracy, when appropriate, will also be investigated. For all examples considered in this section, the boundary conditions are

$$\begin{aligned} \phi & \text{ given on the side AB, BC, CD} \\ P & \text{ given on the side AD} \end{aligned}$$

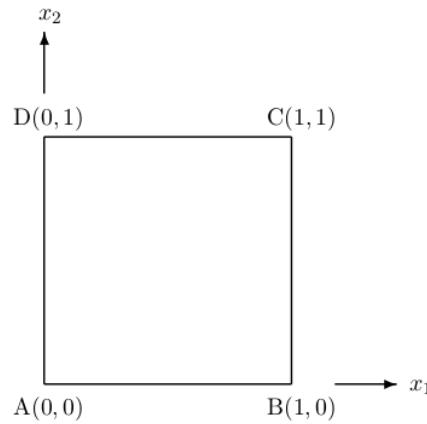


Figure 1. The domain Ω

Table 1. The values of $\bar{\beta}^2$ and corresponding maximum $\beta(\mathbf{x})$ and analytical solutions for Example 1

$\bar{\beta}^2$	max $\beta(\mathbf{x})$	Analytical solution $\phi(\mathbf{x})$
2.4674	4.7124	$\frac{\cos(\frac{\pi}{4}x_1 + \frac{\pi}{4}x_2) + \sin(\frac{\pi}{4}x_1 + \frac{\pi}{4}x_2)}{2(1+0.2x_1+0.3x_2)}$
5.5517	7.0686	$\frac{\cos(\frac{3\pi}{8}x_1 + \frac{3\pi}{8}x_2) + \sin(\frac{3\pi}{8}x_1 + \frac{3\pi}{8}x_2)}{2(1+0.2x_1+0.3x_2)}$
9.8696	9.4248	$\frac{\cos(\frac{\pi}{2}x_1 + \frac{\pi}{2}x_2) + \sin(\frac{\pi}{2}x_1 + \frac{\pi}{2}x_2)}{2(1+0.2x_1+0.3x_2)}$
15.4213	11.7809	$\frac{\cos(\frac{5\pi}{8}x_1 + \frac{5\pi}{8}x_2) + \sin(\frac{5\pi}{8}x_1 + \frac{5\pi}{8}x_2)}{2(1+0.2x_1+0.3x_2)}$

5.1.1. Example 1: Anisotropic quadratically graded material For $k = 0$ one of the possible forms of $g(\mathbf{x})$ satisfying (9) is the quadratic function $g(\mathbf{x}) = [2(1 + 0.2x_1 + 0.3x_2)]^2$ that is when the material under consideration is quadratically graded. The constant coefficient $\bar{\lambda}_{ij}$ is

$$\bar{\lambda}_{ij} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$$

As discussed in [11] and [12], the large wave number $\beta(\mathbf{x})$ is a potential problem in Helmholtz equation because of its impact on the solutions. Therefore we intend to take several values of $\bar{\beta}^2$ and evaluate their effect on the solution errors. The values of $\bar{\beta}^2$ and corresponding maximum value of wave number $\beta(\mathbf{x})$ and analytical solutions are shown in Table 1, where $\pi \approx 3.141592654$.

Table 2 shows convergence of the BEM solutions. Specifically as the total number of elements increases the BEM solutions get closer to the analytical solutions. And Table 3 indicates efficiency of the BEM. Specifically, the standard BEM only needs less than 3.5 minutes time to obtain the solutions $c(\mathbf{x})$ and its derivatives at 19×19 interior points. From this point forward, all the computation results are obtained using total number of 640 elements. Figure 2 shows BEM ϕ absolute errors along the line $x_2 = 0.5$ for several values of $\bar{\beta}^2$. As shown in Figure 2 the standard BEM gives reasonably good ϕ solutions with errors which occur in the third and fourth decimal place. Figure 2 also indicates that in general the errors increase as the wave number gets larger.

Table 2. Convergence of solutions for Example 1 of the case $\bar{\beta}^2 = 2.4674$

Point	160 elements	320 elements	640 elements	Analytical
(0.1,0.5)	0.5751	0.5749	0.5749	0.5748
(0.3,0.5)	0.5772	0.5772	0.5772	0.5772
(0.5,0.5)	0.5656	0.5656	0.5657	0.5657
(0.7,0.5)	0.5412	0.5413	0.5414	0.5414
(0.9,0.5)	0.5055	0.5056	0.5056	0.5056

Table 3. CPU computation time (in seconds) for Example 1 of the case $\bar{\beta}^2 = 2.4674$

160 elements	320 elements	640 elements
4.234375	14.4375	54.671875

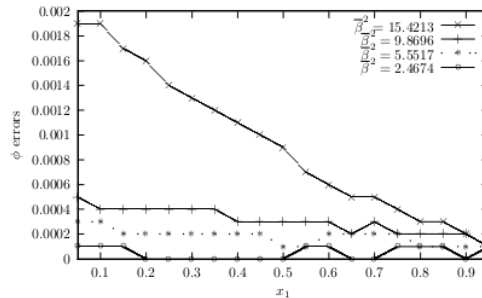


Figure 2. BEM ϕ absolute errors along the line $x_2 = 0.5$ for Example 1

5.1.2. *Example 2: Anisotropic exponentially graded material* When $k < 0$ in equation (9), one of possible forms of $g(\mathbf{x})$ is an exponential function of the form $g(\mathbf{x}) = [2 \exp(2x_1 + 2x_2)]^2$. The constant coefficients $\bar{\lambda}_{ij}$ and k are taken to be

$$\bar{\lambda}_{ij} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix} \quad k = -16$$

We intend to set the value of the coefficient $(k + \bar{\beta}^2)$ in (10) to be negative, zero and positive, so as to consider three different types of equation (10). Therefore we choose $\bar{\beta}^2 = 9, 16, 25$ which implies $k + \bar{\beta}^2 = -7, 0, 9$. Table 4 shows the values of $\bar{\beta}^2$ and corresponding maximum value of wave number $\beta(\mathbf{x})$ and analytical solutions.

Figure 3 shows BEM ϕ absolute errors along the line $x_2 = 0.5$ for three different values of $\bar{\beta}^2$. As shown in Figure 3 the conventional BEM gives good ϕ solutions with errors which occur in the third decimal place, even with very large wave numbers. As for each $\bar{\beta}^2$ represents a different type of equation (10), it is inappropriate to make a conclusion regarding the effect of $\bar{\beta}^2$ values change on the errors.

5.1.3. *Example 3: Anisotropic trigonometrically graded material* Another possible forms of $g(\mathbf{x})$, when $k > 0$ in equation (9), is a trigonometrical function $g(\mathbf{x}) =$

Table 4. The values of $\bar{\beta}^2$ and corresponding maximum $\beta(\mathbf{x})$ and analytical solutions for Example 2

$\bar{\beta}^2$	max $\beta(\mathbf{x})$	Analytical solution $\phi(\mathbf{x})$
9	327.6	$\frac{5 \exp[\sqrt{1.75}(x_1+x_2)]}{2 \exp[2(x_1+x_2)]}$
16	436.8	$\frac{5(1+0.4x_1+0.6x_2)}{2 \exp[2(x_1+x_2)]}$
25	545.9	$\frac{5\{\cos[1.5(x_1+x_2)]+\sin[1.5(x_1+x_2)]\}}{2 \exp[2(x_1+x_2)]}$

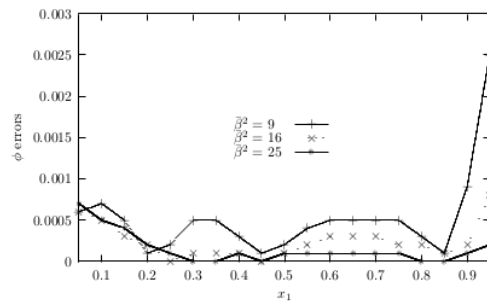


Figure 3. BEM ϕ absolute errors along the line $x_2 = 0.5$ for Example 2

Table 5. The values of $\bar{\beta}^2$ and corresponding analytical solutions for Example 35.1.3

$\bar{\beta}^2$	Analytical solution $\phi(\mathbf{x})$
15	$\frac{5\{\cos[2(x_1+x_2)]+\sin[2(x_1+x_2)]\}}{2\{\cos[0.5(x_1+x_2)]+\sin[0.5(x_1+x_2)]\}}$
8	$\frac{5\{\cos[1.5(x_1+x_2)]+\sin[1.5(x_1+x_2)]\}}{2\{\cos[0.5(x_1+x_2)]+\sin[0.5(x_1+x_2)]\}}$
3	$\frac{5\{\cos(x_1+x_2)+\sin(x_1+x_2)\}}{2\{\cos[0.5(x_1+x_2)]+\sin[0.5(x_1+x_2)]\}}$

$[2\{\cos(0.5x_1 + 0.5x_2) + \sin(0.5x_1 + 0.5x_2)\}]^2$. We take the constant coefficients $\bar{\lambda}_{ij}$ and k

$$\bar{\lambda}_{ij} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix} \quad k = 1$$

The value of the coefficient $(k + \bar{\beta}^2)$ in (10) is always positive. Therefore we only have one type of equation (10). Table 5 shows the values of $\bar{\beta}^2$ and corresponding analytical solutions.

Again, it is observed from Figure 4 that the errors of BEM ϕ solutions for increase as the value of $\bar{\beta}^2$ gets larger.

5.2. Problems without simple analytical solutions

In this section two problems without any simple analytical solutions will be considered. The purposes are to show coherence between the flow vector $\left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right)$ and the scattering ϕ solutions inside the domain, the impact of the inhomogeneity (homogeneous or inhomogeneous) and the anisotropy (isotropic or anisotropic) of the material, and also to see comparison of solutions for quadratically, exponentially and trigonometrically graded materials by keeping the parameters A, α_m of the function $g(\mathbf{x})$, boundary conditions, and the constant coefficients $\bar{\lambda}_{ij}, \bar{\beta}^2$ the same for all types of graded materials.

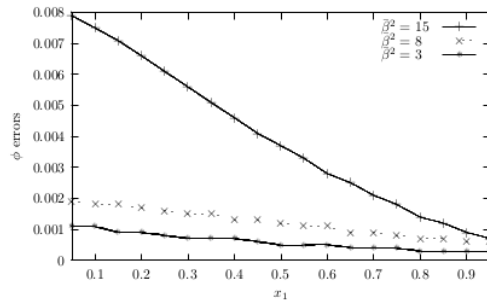


Figure 4. BEM ϕ absolute errors along the line $x_2 = 0.5$ for Example 3

Table 6. Two cases regarding the anisotropy ($\bar{\lambda}_{ij}$) and inhomogeneity ($g(\mathbf{x})$) for Example 4

Material	$\bar{\lambda}_{ij}$	$g(\mathbf{x})$
isotropic homogeneous	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	4
anisotropic inhomogeneous	$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$	$[2(1 + 0.2x_1 + 0.3x_2)]^2$

5.2.1. Example 4 Now, we seek a BEM solution to an interior BVP governed by the Helmholtz type equation (1) inside the domain as depicted in Figure 1 with boundary conditions

$$\begin{aligned} P &= 0 && \text{on the side AB} \\ \phi &= 0 && \text{on the side BC} \\ P &= 0 && \text{on the side CD} \\ P &= 1 && \text{on the side AD} \end{aligned}$$

The variable coefficients $\lambda_{ij}(\mathbf{x})$ and $\beta^2(\mathbf{x})$ for the governing equation (1) are

$$\begin{aligned} \lambda_{ij}(\mathbf{x}) &= \bar{\lambda}_{ij}g(\mathbf{x}) \\ \beta^2(\mathbf{x}) &= \bar{\beta}^2g(\mathbf{x}) \\ g(\mathbf{x}) &= [2(\alpha_0 + \alpha_1x_1 + \alpha_2x_2)]^2 \\ \bar{\beta}^2 &= 40 \end{aligned}$$

And we consider two cases regarding the anisotropy ($\bar{\lambda}_{ij}$) and inhomogeneity ($g(\mathbf{x})$) of the material as shown in Table 6. Figures 5 – 6 show coherence between the flow $\left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}\right)$ and scattering ϕ solutions. This verifies that the developed FORTRAN code computes the flow vector $\left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}\right)$ correctly.

5.2.2. Example 5 A comparison of ϕ solutions inside the domain for quadratically, exponentially and trigonometrically graded materials, by conserving the parameters A, α_m of the function $g(\mathbf{x})$, the boundary conditions, and the constant coefficients $\bar{\lambda}_{ij}, \bar{\beta}^2$ for all the types of graded material, will be shown. Three types of material's gradation and their forms of function $g(\mathbf{x})$ are

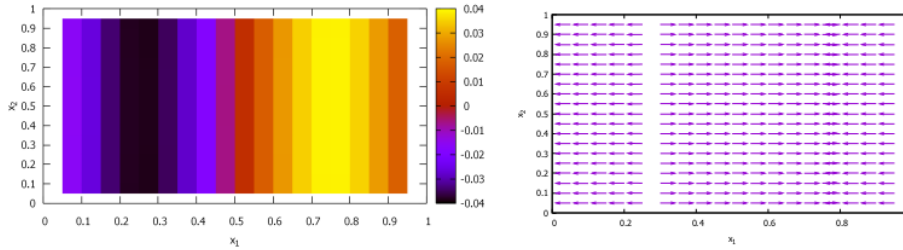


Figure 5. The scattering and flow solutions for Example 4 of the isotropic homogeneous material

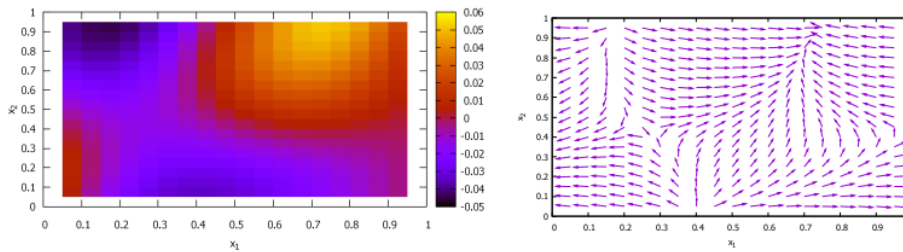


Figure 6. The scattering and flow solutions for Example 4 of the anisotropic inhomogeneous material

$$\begin{aligned}
 \text{quadratical} \quad & g(\mathbf{x}) = [A(\alpha_0 + \alpha_m x_m)]^2, \quad k = 0 \\
 \text{exponential} \quad & g(\mathbf{x}) = [A \exp(\alpha_m x_m)]^2, \quad k = -\bar{\lambda}_{ij} \alpha_i \alpha_j \\
 \text{trigonometrical} \quad & g(\mathbf{x}) = [A \{\cos(\alpha_m x_m) + \sin(\alpha_m x_m)\}]^2, \quad k = \bar{\lambda}_{ij} \alpha_i \alpha_j
 \end{aligned}$$

The parameter $\bar{\beta}^2$ is chosen to be $\bar{\beta}^2 = 40$ and the boundary conditions are

$$\begin{aligned}
 P &= 0 && \text{on the side AB} \\
 \phi &= 0 && \text{on the side BC} \\
 P &= 0 && \text{on the side CD} \\
 P &= 10 && \text{on the side AD}
 \end{aligned}$$

and the values of constant matrix $\bar{\lambda}_{ij}$ and the parameters α_m associated with the anisotropy and inhomogeneity of the material are shown in Table 7.

Table 8 shows a comparison of ϕ solutions inside the domain for each combination of isotropy and homogeneity, and each type of types of material's gradation. The results in Table 8 may be described as follows:

- for each type of material, the impact of the anisotropy and inhomogeneity on the solutions is evident. This suggests that it is important to take into account the anisotropy as well as the inhomogeneity in application.
- when the material is homogeneous (ie. $\alpha_1 = 0, \alpha_2 = 0$ so that $k = 0$), either the material is isotropic or anisotropic, all the three types of material give identical solutions since the problems are identical

Table 7. The values of constant matrix $\bar{\lambda}_{ij}$ and the parameters α_m for Example 5

Material	$\bar{\lambda}_{ij}$	$g(\mathbf{x})$
isotropic homogeneous	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$A = 2, \alpha_0 = 1, \alpha_1 = 0, \alpha_2 = 0$
isotropic inhomogeneous	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$A = 2, \alpha_0 = 1, \alpha_1 = \frac{7\pi}{16}, \alpha_2 = \frac{7\pi}{16}$
anisotropic homogeneous	$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$	$A = 2, \alpha_0 = 1, \alpha_1 = 0, \alpha_2 = 0$
anisotropic inhomogeneous	$\begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}$	$A = 2, \alpha_0 = 1, \alpha_1 = \frac{7\pi}{16}, \alpha_2 = \frac{7\pi}{16}$

- vice versa, when the material is inhomogeneous (ie. $\alpha_1 = \frac{7\pi}{16}, \alpha_2 = \frac{7\pi}{16}$) the scattering solutions of the three types of material's gradation are different. This is due to that the problems are not identical (the value k in equations (9) and (10) is different) for each type of material's gradation.

6. Conclusion

It is certainly possible to use a standard BEM for solving BVPs governed by an equation of variable coefficients such as the Helmholtz type equation (1). One way to do it, which is adopted in this work, is by transforming the variable coefficient equation into a constant coefficient equation from which a boundary integral equation can be derived. This boundary integral equation becomes a starting point for constructing a BEM. Implementation of the standard BEM is rather easy and the numerical solution resulted from it is sufficiently accurate and timeless.

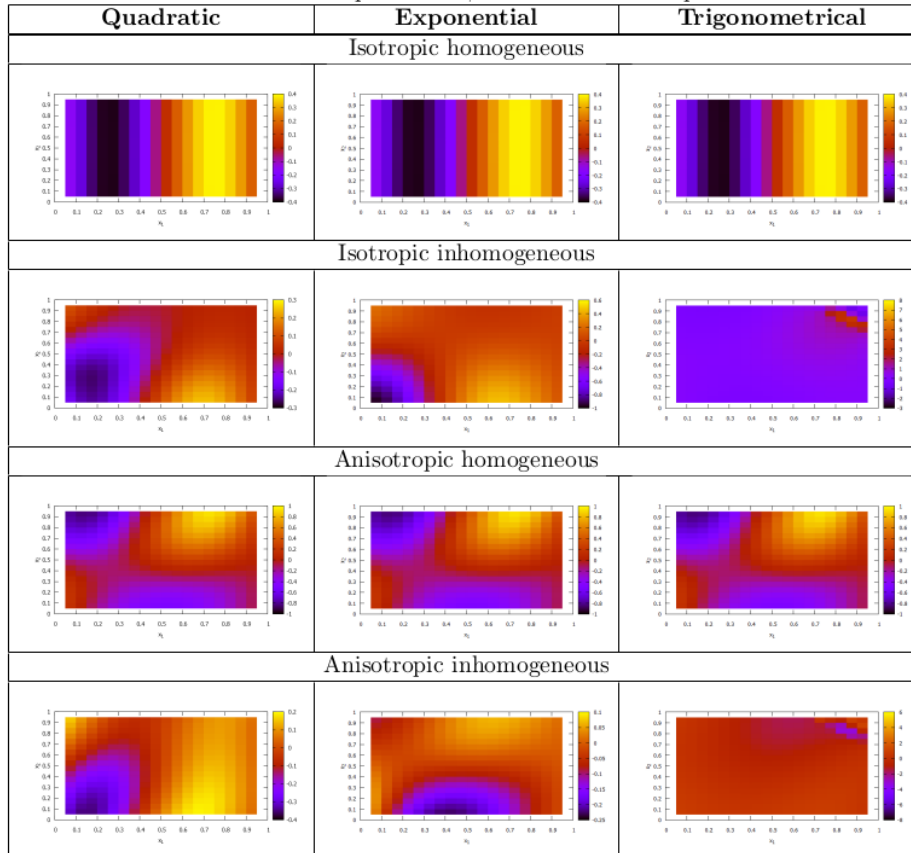
A variable coefficients governing equation such as (1) is usually used for modelling physical application for an anisotropic FGM. In this paper, types of FGMs covered are quadratically, exponentially and trigonometrically graded materials.

In addition to its accuracy (even with large wave numbers) and efficiency (short computation time), consistency between the flow vectors and scattering solutions of the BEM have been shown so as to say the BEM has been working appropriately. Moreover, it is also observed that the anisotropy and inhomogeneity of the material effect the results. This suggests both anisotropy and inhomogeneity should be taken into account in applications.

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Table 8. Comparison of ϕ solutions for Example 5



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